Weak amenability for dynamical systems

Citation for the original published paper (version of record):

McKee, A. (2021)
Weak amenability for dynamical systems
Studia Mathematica, 258(1): 53-70
http://dx.doi.org/10.4064/SM200227-20-7

STUDIA MATHEMATICA 258 (1) (2021)

Weak amenability for dynamical systems

by

Andrew McKee (Göteborg)

Abstract. Using the recently developed notion of a Herz–Schur multiplier of a C^*-dynamical system we introduce weak amenability of C^*- and W^*-dynamical systems. As a special case we recover Haagerup's characterisation of weak amenability of a discrete group. We also consider a generalisation of the Fourier algebra and its multipliers to crossed products.

1. Introduction. Among the many characterisations of amenability of a locally compact group G is Leptin's Theorem [12]: G is amenable if and only if the Fourier algebra of G has a bounded approximate identity. The idea to weaken the latter condition, by requiring the approximate identity to be bounded in a different norm, goes back to Haagerup [9]. Following this, Cowling–Haagerup [5] formally defined weak amenability, explored some equivalent conditions, and introduced the Cowling–Haagerup (or weak amenability) constant. This constant has been computed for a large number of groups: see Brown–Ozawa [4, Theorem 12.3.8] and the references given by Knudby [11]. An overview of the literature surrounding weak amenability can be found in [11, Section 5].

Weak amenability is an example of a property defined in terms of functions on a group which can be characterised by an approximation property of the group von Neumann algebra and/or group C^*-algebra (see [4, Chapter 12] for several examples of such properties); the aim of this paper is to extend this idea to crossed products. A C^*-algebra A is said to have the *completely bounded approximation property* (CBAP) if there exists a net (T_γ) of finite rank completely bounded maps on A such that $T_\gamma \to \mathrm{id}_A$ in the point-norm topology and $\sup_\gamma \|T_\gamma\|_{\mathrm{cb}} = C < \infty$. The infimum of all such constants C is denoted $\Lambda_{\mathrm{cb}}(A)$. Similarly, a von Neumann algebra M is said

2020 *Mathematics Subject Classification*: Primary 46L55; Secondary 46L05.
Key words and phrases: Schur multiplier, C^*-crossed products, approximation properties, weak amenability.
Received 27 February 2020; revised 1 July 2020.
Published online 15 October 2020.

DOI: 10.4064/sm200227-20-7

to have the *weak* completely bounded approximation property* (weak* CBAP) if there exists a net (R_γ) of ultraweakly continuous, finite rank, completely bounded maps on M such that $R_\gamma \to \mathrm{id}_M$ in the point-weak* topology and $\sup_\gamma \|R_\gamma\|_{\mathrm{cb}} = C < \infty$; again, the infimum of all such constants C is denoted $\Lambda_{\mathrm{cb}}(M)$. A locally compact group G is called *weakly amenable* if there exists a net of compactly supported Herz–Schur multipliers on G, uniformly bounded in the Herz–Schur multiplier norm, converging uniformly to 1 on compact sets. Haagerup [9, Theorem 2.6] proved that a discrete group is weakly amenable if and only if the reduced group C^*-algebra has the completely bounded approximation property, if and only if the group von Neumann algebra has the weak* completely bounded approximation property.

In this paper we define weak amenability of C^*- and W^*-dynamical systems and characterise a weakly amenable system in terms of the completely bounded approximation property of the corresponding crossed product. The results in this direction, Theorems 4.3 and 4.6, may be seen as a generalisation of Haagerup's result above. Haagerup–Kraus [10, Section 3] have studied W^*-dynamical systems under the assumption that G is weakly amenable; Proposition 4.8 was motivated by their Theorem 3.2(b) and Remark 3.10.

In Section 2 we review the definitions and results surrounding the notion of a Herz–Schur multiplier of a C^*-dynamical system. Section 3 is motivated by the description of Herz–Schur multipliers as completely bounded multipliers of the Fourier algebra; we view the predual of (the enveloping von Neumann algebra of) the reduced crossed product as consisting of vector-valued functions on the group, and describe the completely bounded multipliers of this space as certain Herz–Schur multipliers of the associated dynamical system. In Section 4 we define weak amenability of C^*- and W^*-dynamical systems, and provide a characterisation.

2. Preliminaries. In this section we review the definitions and results of [13] required later, as well as establish notation. Throughout, G will denote a second-countable, locally compact, topological group, endowed with left Haar measure m; integration on G, with respect to m, over the variable s is simply denoted ds. Write λ^G for the left regular representation of G on $L^2(G)$, and for the corresponding representation of $L^1(G)$. The reduced group C^*-algebra $C_r^*(G)$ and group von Neumann algebra $\mathrm{vN}(G)$ of G are, respectively, the closure of $\lambda^G(L^1(G))$ in the norm and weak* topology of $\mathcal{B}(L^2(G))$; we also have $\mathrm{vN}(G) = \{\lambda_s^G : s \in G\}''$.

Throughout, A will be a unital, separable C^*-algebra. Let $\alpha : G \to \mathrm{Aut}(A)$ be a group homomorphism which is continuous in the point-norm topology, i.e. for all $a \in A$ the map $s \mapsto \alpha_s(a)$ is continuous from G to A; in short, consider a C^*-dynamical system (A, G, α).

Let θ be a faithful representation of A on $\mathcal{H}_\theta$ and define representations of A and G on $L^2(G, \mathcal{H}_\theta)$ by

$$(\pi^\theta(a)\xi)(s) := \theta(\alpha_{s^{-1}})(a)(\xi(s)), \qquad (\lambda_t^\theta \xi)(s) := \xi(t^{-1}s),$$

for all $a \in A$, $s, t \in G$, $\xi \in L^2(G, \mathcal{H}_\theta)$. It is easy to check that

$$\pi^\theta(\alpha_t(a)) = \lambda_t^\theta \pi^\theta(a)(\lambda_t^\theta)^*, \qquad a \in A,\ t \in G.$$

The pair $(\pi^\theta, \lambda^\theta)$ is therefore a *covariant representation* of (A, G, α). Thus we obtain a representation $\pi^\theta \rtimes \lambda^\theta$ of the Banach $*$-algebra $L^1(G, A)$ on $L^2(G, \mathcal{H}_\theta)$ given by

$$\pi^\theta \rtimes \lambda^\theta(f) := \int_G \pi^\theta(f(s))\lambda_s^\theta \, ds, \qquad f \in L^1(G, A).$$

The *reduced crossed product* of A by G is defined as the closure of $(\pi^\theta \rtimes \lambda^\theta)(L^1(G, A))$ in the operator norm of $\mathcal{B}(L^2(G, \mathcal{H}_\theta))$; it does not depend on the choice of faithful representation θ so we will often omit the superscript θ from our notation, and denote the reduced crossed product by $A \rtimes_{\alpha,r} G$, writing $A \rtimes_{\alpha,\theta} G$ when we wish to emphasise the choice of θ. The *full crossed product* of A by G, denoted $A \rtimes_\alpha G$, is the C^*-algebra obtained by completing $L^1(G, A)$ in the universal norm

$$\|f\| := \sup\{\|\rho \rtimes \tau(f)\| : (\rho, \tau) \text{ is a covariant representation of } (A, G, \alpha)\}.$$

We refer to Pedersen [14, Chapter 7] and Williams [18] for the details of these constructions.

In [13] the present author, with Todorov and Turowska, introduced and studied Herz–Schur multipliers of a C^*-dynamical system, extending the classical notion of a Herz–Schur multiplier (see De Cannière–Haagerup [6]). We now recall the definitions and results needed here; the classical definitions of Herz–Schur multipliers are the special case $A = \mathbb{C}$ of the definitions below. A bounded function $F : G \to \mathcal{B}(A)$ will be called *pointwise measurable* if, for every $a \in A$, the map $s \mapsto F(s)(a)$ is a weakly measurable function from G to A. For each $f \in L^1(G, A)$ define $F \cdot f(s) := F(s)(f(s))$ $(s \in G)$. If F is bounded and pointwise measurable then $F \cdot f$ is weakly measurable and $\|F \cdot f\|_1 \le \sup_{s\in G} \|F(s)\| \, \|f\|_1$, so $F \cdot f \in L^1(G, A)$ for every $f \in L^1(G, A)$. We write $\mathcal{CB}(A)$ for the collection of completely bounded maps on A.

Definition 2.1. Let $\theta : A \to \mathcal{B}(\mathcal{H}_\theta)$ be a faithful representation of A on a separable Hilbert space. A bounded, pointwise measurable function $F : G \to \mathcal{CB}(A)$ will be called a *Herz–Schur (A, G, α)-multiplier* if the map $S_F^\theta : (\pi^\theta \rtimes \lambda^\theta)(L^1(G, A)) \to (\pi^\theta \rtimes \lambda^\theta)(L^1(G, A))$ given by

$$S_F^\theta((\pi^\theta \rtimes \lambda^\theta)(f)) := (\pi^\theta \rtimes \lambda^\theta)(F \cdot f)$$

is completely bounded; if this is the case then S_F^θ has a unique extension to a completely bounded map on $A \rtimes_{\alpha,r} G$. The set of all Herz–Schur (A, G, α)-

multipliers is an algebra with respect to the obvious operations; we denote it by $\mathfrak{S}(A,G,\alpha)$ and endow it with the norm $\|F\|_{\mathrm{HS}} := \|S_F\|_{\mathrm{cb}}$.

The above definition does not depend on the faithful representation θ [13, Remark 3.2(ii)]. Let $\alpha^\theta : G \to \mathrm{Aut}(\theta(A))$ be given by $\alpha^\theta_t(\theta(a)) := \theta(\alpha_t(a))$ $(t \in G,\ a \in A)$; note that if α is continuous in the point-norm topology then so is α^θ. We say α is a *θ-action* if, for every $t \in G$, α^θ_t extends to a weak*-continuous automorphism of $\theta(A)''$ such that the map $t \mapsto \alpha^\theta_t(x)$ is weak*-continuous for each $x \in \theta(A)''$. We will need to work with $\overline{A \rtimes_{\alpha,\theta} G}^{\mathrm{w}^*}$, which we denote by $A \rtimes^{\mathrm{w}^*}_{\alpha,\theta} G$.

Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$, and $\beta : G \to \mathrm{Aut}(M)$ a group homomorphism which is continuous in the point-weak* topology; then the triple (M,G,β) is called a *W^*-dynamical system.* Defining representations π and λ of M and G respectively on $L^2(G,\mathcal{H})$ by the same formulae as above gives a covariant pair of representations (π,λ) of (M,G,β), with π normal. The (*von Neumann*) *crossed product* of (M,G,β), denoted $M \rtimes^{\mathrm{vN}}_\beta G$, is the von Neumann algebra generated by $\pi(M)$ and $\lambda(G)$ on $L^2(G,\mathcal{H})$. See Takesaki [17, Chapter X] for more on this construction.

Classically, $u : G \to \mathbb{C}$ is called a Herz–Schur multiplier if u is a completely bounded multiplier of the Fourier algebra of G (the Fourier algebra of G, $A(G)$, will be defined in Section 3), i.e. $uv \in A(G)$ for all $v \in A(G)$ and the map

$$m_u : A(G) \to A(G), \quad m_u(v) := uv, \quad v \in A(G),$$

is completely bounded; the space of such functions is denoted $\mathrm{M}^{\mathrm{cb}}A(G)$. Bożejko–Fendler [3] discuss several equivalent definitions of Herz–Schur multipliers, including: Herz–Schur multipliers on G coincide with the completely bounded multipliers of $\mathrm{vN}(G)$. One can further show that if u is a Herz–Schur multiplier of G then $m_u^* : \mathrm{vN}(G) \to \mathrm{vN}(G)$ leaves $C^*_r(G)$ invariant. In defining Herz–Schur (A,G,α)-multipliers we took the reverse approach, defining first a map on $A \rtimes_{\alpha,r} G$. If the dynamical system in question is $(\mathbb{C},G,1)$ then the corresponding crossed product is precisely $C^*_r(G)$, so (identifying $\mathcal{CB}(\mathbb{C})$ with $\mathbb{C}$) we see that u is a Herz–Schur $(\mathbb{C},G,1)$-multiplier if and only if it is a Herz–Schur multiplier. The goal of Section 3 is to introduce a space for a C^*-dynamical system (A,G,α) which generalises the Fourier algebra of a locally compact group, and identify Herz–Schur (A,G,α)-multipliers with the completely bounded 'multipliers' of this space. In contrast to the classical case it is not clear if the map S_F corresponding to $F \in \mathfrak{S}(A,G,\alpha)$ extends to the weak*-closure of $A \rtimes_{\alpha,r} G$, so we make the following definition.

Definition 2.2. Let $(\theta,\mathcal{H}_\theta)$ be a faithful representation of A on a separable Hilbert space. A bounded function $F : G \to \mathcal{CB}(A)$ will be called a

Herz–Schur θ-multiplier of (A, G, α) if the map

$$S_F^\theta : \pi^\theta(a)\lambda_t^\theta \mapsto \pi^\theta(F(t)(a))\lambda_t^\theta, \qquad a \in A,\, t \in G,$$

has an extension to a completely bounded weak*-continuous map on $A \rtimes_{\alpha,\theta}^{\mathrm{w}^*} G$.

Observe that [13, Remark 3.4] shows that Herz–Schur θ-multipliers of (A, G, α) act in the same way as Herz–Schur (A, G, α)-multipliers, when viewed through a weak*-continuous functional. To simplify notation we will often omit the superscript θ from the maps S_F associated to the multipliers defined above; it will be clear from the presence/absence of θ elsewhere in the notation where S_F is acting.

The following result [13, Theorem 3.8] provides a useful characterisation of Herz–Schur (A, G, α)-multipliers, generalising the classical transference theorem (see e.g. [3]).

THEOREM 2.3. *Let (A, G, α) be a C^*-dynamical system with $A \subseteq \mathcal{B}(\mathcal{H})$, and let $F : G \to \mathcal{CB}(A)$ be a bounded, pointwise measurable function. The following are equivalent:*

(i) *F is a Herz–Schur (A, G, α)-multiplier;*
(ii) *there exist a separable Hilbert space $\mathcal{H}_\rho$, a non-degenerate representation $\rho : A \to \mathcal{B}(\mathcal{H}_\rho)$ and $V, W \in L^\infty(G, \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho))$ such that*

$$\mathcal{N}(F)(s,t)(a) := \alpha_{t^{-1}}\big(F(ts^{-1})(\alpha_t(a))\big) = W(t)^*\rho(a)V(s).$$

REMARK 2.4. Bédos and Conti [1, Section 4] have taken a Hilbert C^*-module approach to completely bounded multipliers of a discrete (twisted) C^*-dynamical system. It is easy to check that $F : G \to \mathcal{CB}(A)$ is a Herz–Schur (A, G, α)-multiplier if and only if $T_F : G \times A \to A$, $T_F(t,a) := F(t)(a)$ ($t \in G$, $a \in A$), is a completely bounded reduced multiplier of (A, G, α), in the sense of Bédos–Conti. The same authors have also introduced a version of the Fourier–Stieltjes algebra for discrete (twisted) C^*-dynamical systems, again using Hilbert C^*-modules [2].

3. Fourier space of a dynamical system. In this section we develop a space for the crossed product which is analogous to the Fourier algebra in the setting of group C^*-algebras and von Neumann algebras, and study the multipliers of this space. To motivate this discussion and fix notation let us first recall some facts about the Fourier algebra of a locally compact group G. The *Fourier algebra* of G, introduced by Eymard [7], denoted $A(G)$, is the space of coefficients of the left regular representation, that is, the space of functions $u : G \to \mathbb{C}$ of the form

$$u(t) = \langle \lambda_t^G \xi, \eta \rangle, \qquad t \in G,\, \xi, \eta \in L^2(G).$$

The linear space defined in this way becomes an algebra under pointwise multiplication, and turns out to be the predual of the group von Neumann algebra $\mathrm{vN}(G)$. Bożejko–Fendler [3] proved that the space $\mathrm{M}^{\mathrm{cb}}A(G)$ is isometrically isomorphic to the space of Herz–Schur multipliers of G, so they are treated as the same space.

Recall that A denotes a unital C^*-algebra and $\alpha : G \to \mathrm{Aut}(A)$ is a point-norm continuous homomorphism. The following definition is adapted from Pedersen [14, 7.7.4].

Definition 3.1. Let (A, G, α) be a C^*-dynamical system and let $(\theta, \mathcal{H}_\theta)$ be a faithful representation of A. Let $\tilde{u} \in (A \rtimes_{\alpha,\theta} G)^*$ be a functional of the form

$$\tilde{u}(T) = \sum_{n\in\mathbb{N}} \langle T\xi_n, \eta_n\rangle, \qquad T \in A \rtimes_{\alpha,\theta} G, \tag{1}$$

where $\xi_n, \eta_n \in L^2(G, \mathcal{H}_\theta)$ satisfy $\sum_n \|\xi_n\|^2 < \infty$ and $\sum_n \|\eta_n\|^2 < \infty$. The set of such functionals forms a linear space which can be identified with $((A \rtimes_{\alpha,\theta} G)'')_*$. To each such $\tilde{u}$ we associate the function $u : G \to A^*$ defined by

$$u(t)(a) := \tilde{u}(\pi^\theta(a)\lambda_t^\theta), \qquad a \in A,\ t \in G. \tag{2}$$

The set of all functions from G to A^* associated to functionals of the form of $\tilde{u}$ is a linear space (with the obvious operations), which we again identify with the predual of $(A \rtimes_{\alpha,\theta} G)''$ and endow with the norm

$$\|u\|_{\mathcal{A}} := \|\tilde{u}\|,$$

where the right side means the norm of $\tilde{u}$ as a member of the dual space of $(A \rtimes_{\alpha,\theta} G)''$. The resulting space is called the *Fourier space* of (A, G, α) and denoted $\mathcal{A}^\theta(A, G, \alpha)$.

In the case of the system $(\mathbb{C}, G, 1)$, if we let θ be the one-dimensional representation of $\mathbb{C}$ then $\pi^\theta = \theta \otimes \mathrm{id}$, and we can identify λ^θ with λ^G; thus the above definition gives the predual of $(\mathbb{C} \rtimes_{1,r} G)'' \cong \mathrm{vN}(G)$, so the space defined may be identified with $A(G)$. Definition 3.1 also works unchanged for a W^*-dynamical system (M, G, β); in this case the definition identifies the predual of the von Neumann algebra $M \rtimes_\beta^{\mathrm{vN}} G$ with the space of functions $u : G \to M_*$ corresponding to functionals of the form (1) [16]. The following is shown by Fujita [8, Lemma 3.4].

Remark 3.2. Let (A, G, α) be a C^*-dynamical system and $(\theta, \mathcal{H}_\theta)$ a faithful representation of A. The compactly supported functions form a dense subset of $\mathcal{A}^\theta(A, G, \alpha)$. The same holds for a W^*-dynamical system.

It appears that the space $\mathcal{A}^\theta(A, G, \alpha)$ was first defined for W^*-dynamical systems and their crossed products by Takai [16]. Note that in the case of a W^*-dynamical system Fujita [8] introduces a Banach algebra structure on $\mathcal{A}^\theta(A, G, \alpha)$, but we do not pursue this here.

We now define multipliers of the Fourier space of a C^*-dynamical system, and study the relationship with Herz–Schur multipliers of the system. The results in this section are essentially predual versions of some results in [13, Section 3].

DEFINITION 3.3. A bounded function $F : G \to \mathcal{B}(A)$ is called a *multiplier of* $\mathcal{A}^\theta(A, G, \alpha)$ if there is a bounded map

$$s_F : \mathcal{A}^\theta(A, G, \alpha) \ \to \mathcal{A}^\theta(A, G, \alpha)$$

such that

$$(s_F u)(t)(a) = u(t)(F(t)(a)), \qquad u \in \mathcal{A}^\theta(A, G, \alpha),\, t \in G,\, a \in A.$$

The norm of a multiplier F is defined by $\|F\|_{\mathrm{M}} := \|s_F^*\|$. If moreover F maps into $\mathcal{CB}(A)$ and the dual map s_F^* is completely bounded then F is called a *completely bounded multiplier* of $\mathcal{A}^\theta(A, G, \alpha)$. In this case the completely bounded multiplier norm of F is defined by $\|F\|_{\mathrm{M}^{\mathrm{cb}}} := \|s_F^*\|_{\mathrm{cb}}$. The spaces of bounded and completely bounded multipliers of $\mathcal{A}^\theta(A, G, \alpha)$ are denoted $\mathrm{M}\mathcal{A}^\theta(A, G, \alpha)$ and $\mathrm{M}^{\mathrm{cb}}\mathcal{A}^\theta(A, G, \alpha)$ respectively.

LEMMA 3.4. *Let $F : G \to \mathcal{B}(A)$ be a bounded, pointwise measurable function, and $(\theta, \mathcal{H}_\theta)$ be a faithful representation of A. The following are equivalent:*

(i) *F is a multiplier of $\mathcal{A}^\theta(A, G, \alpha)$;*
(ii) *there is an ultraweakly continuous bounded operator S_F on $(A \rtimes_{\alpha,\theta} G)''$ such that $S_F(\pi^\theta(a)\lambda_t^\theta) = \pi^\theta(F(t)(a))\lambda_t^\theta$ for all $a \in A,\ t \in G$.*

Moreover, if either condition holds then $\|F\|_{\mathrm{M}} = \|S_F\|$. Finally, F is a completely bounded multiplier of $\mathcal{A}^\theta(A, G, \alpha)$ if and only if the map S_F of (ii) *is completely bounded, and in this case $\|F\|_{\mathrm{M}^{\mathrm{cb}}} = \|S_F\|_{\mathrm{cb}}$.*

Proof. If F is a multiplier of $\mathcal{A}^\theta(A, G, \alpha)$ then $S_F := s_F^*$ is the required map because for any $u \in \mathcal{A}^\theta(A, G, \alpha)$,

$$\langle S_F(\pi^\theta(a)\lambda_t^\theta), u\rangle = \langle \pi^\theta(a)\lambda_t^\theta, s_F u\rangle = u(t)(F(t)(a)) = \langle \pi^\theta(F(t)(a))\lambda_t^\theta, u\rangle.$$

Conversely, given $u \in \mathcal{A}^\theta(A, G, \alpha)$, the function

$$\pi^\theta(a)\lambda_t^\theta \mapsto \langle S_F(\pi^\theta(a)\lambda_t^\theta), u\rangle$$

extends to an ultraweakly continuous linear functional on $(A \rtimes_{\alpha,\theta} G)''$. Therefore, there is $Fu \in \mathcal{A}^\theta(A, G, \alpha)$, with $\|Fu\| \le \|u\|_{\mathcal{A}}\|S_F\|$, such that $\langle \pi^\theta(a)\lambda_t^\theta, Fu\rangle = \langle S_F(\pi^\theta(a)\lambda_t^\theta), u\rangle$. It follows that the map $u \mapsto Fu$ is continuous, and

$$(Fu)(t)(a) = \langle \pi^\theta(a)\lambda_t^\theta, Fu\rangle = \langle S_F(\pi^\theta(a)\lambda_t^\theta), u\rangle = u(t)(F(t)(a))$$

for all $t \in G,\ a \in A$, so F is a multiplier of $\mathcal{A}^\theta(A, G, \alpha)$ with $s_F u = Fu$ for all $u \in \mathcal{A}^\theta(A, G, \alpha)$. Finally, $\|F\|_M = \|s_F^*\| = \|S_F\|$ by definition.

The statements about completely bounded multipliers follow similarly. ∎

Since the ultraweak topology on $(A \rtimes_{\alpha,\theta} G)''$ is the relative ultraweak topology from $\mathcal{B}(L^2(G) \otimes \mathcal{H}_\theta)$, we consider the map S_F of the previous lemma to be a weak*-continuous map on $A \rtimes_{\alpha,\theta}^{\mathrm{w}^*} G$.

COROLLARY 3.5. *The space of Herz–Schur θ-multipliers of (A, G, α) coincides isometrically with the space of completely bounded multipliers of $\mathcal{A}^\theta(A, G, \alpha)$.*

Proof. Immediate from Lemma 3.4 and [13, Corollary 3.10]. ∎

In the next section we will use the description of Herz–Schur multipliers of a dynamical system as completely bounded multipliers of the Fourier space in studying weak amenability of the system.

4. Weak amenability. In this section we define weak amenability of a C^*-dynamical system; when the group is discrete, we show this is equivalent to the CBAP of the reduced crossed product. We also define weak amenability of a W^*-dynamical system, and when the group is discrete and weakly amenable, we show this is equivalent to the weak* CBAP of the associated crossed product. The weak* CBAP for crossed products of W^*-dynamical systems has been studied by Haagerup–Kraus [10, Section 3]; they showed that if (M, G, α) is a W^*-dynamical system with G weakly amenable and M having the weak* CBAP then it is not true in general that $M \rtimes_\alpha^{\mathrm{vN}} G$ has the weak* CBAP. The CBAP for the reduced crossed product of a C^*-dynamical system has been studied by Sinclair–Smith [15] under the assumption that the group is amenable; here we give some other conditions under which the reduced crossed product has the CBAP.

As before, A is a unital C^*-algebra and $(\theta, \mathcal{H}_\theta)$ is a faithful representation of A. In this section, G will always denote a discrete group. Denote by $\alpha : G \to \mathrm{Aut}(A)$ a homomorphism, so that (A, G, α) is a C^*-dynamical system. Since G is discrete, there is a canonical conditional expectation $\mathcal{E}^\theta : \theta(A) \rtimes_{\alpha^\theta, r} G \to \theta(A)$ which is equivariant (see Brown–Ozawa [4, Proposition 4.1.9]). We denote by $\mathcal{E}$ the completely positive map defined by

$$A \rtimes_{\alpha,\theta} G \cong \theta(A) \rtimes_{\alpha^\theta, r} G \to A, \qquad \sum_{t \in G} \pi^\theta(a_t)\lambda_t^\theta \mapsto a_e, \quad a_t \in A.$$

The triple (M, G, β) will denote a discrete W^*-dynamical system, i.e. M is a von Neumann algebra acting on a Hilbert space $\mathcal{H}_M$, G is a discrete group, and $\beta : G \to \mathrm{Aut}(M)$ is a homomorphism. The symbol $\mathcal{E}$ will also be used for the conditional expectation $M \rtimes_\beta^{\mathrm{vN}} G \to M$, defined similarly.

Our main questions are:

- For a C^*-dynamical system (A, G, α), what are necessary and/or sufficient conditions for $A \rtimes_{\alpha,\theta} G$ to have the completely bounded approximation property?

- For a W^*-dynamical system (M, G, β), what are necessary and/or sufficient conditions for $M \rtimes_{\beta}^{\mathrm{vN}} G$ to have the weak* completely bounded approximation property?

Our approach to these problems is to consider certain Herz–Schur multipliers of the system in question. Since we have so far only considered Herz–Schur multipliers of a C^*-dynamical system, we briefly describe a construction, mentioned by Fujita [8, p. 56], which shows that Herz–Schur multipliers of a W^*-dynamical system are particular cases of the weak*-extendable multipliers of Definition 2.2. For the W^*-dynamical system (M, G, β), where M is a von Neumann algebra on the separable Hilbert space $\mathcal{H}_M$, consider the set

$$M_\beta := \{x \in M : t \mapsto \beta_t(x) \text{ is norm continuous}\}.$$

Then M_β is a G-invariant, weak*-dense C^*-subalgebra of M containing the identity, and (M_β, G, β) is a C^*-dynamical system, with M_β faithfully represented on $\mathcal{B}(\mathcal{H}_M)$. The construction of the reduced crossed product $M_\beta \rtimes_{\beta,r} G$, using the faithful representation $\mathrm{id} : M_\beta \to \mathcal{B}(\mathcal{H}_M)$, gives a weak*-dense C^*-subalgebra of $M \rtimes_{\beta}^{\mathrm{vN}} G$. It follows that $\mathcal{A}^{\mathrm{id}}(M_\beta, G, \beta)$ can be identified with the predual of $M \rtimes_{\beta}^{\mathrm{vN}} G$, and that the Herz–Schur id-multipliers of (M_β, G, β) are completely bounded multipliers of $\mathcal{A}^{\mathrm{id}}(M_\beta, G, \beta)$, and the associated maps possess completely bounded, weak*-continuous extensions to $M \rtimes_{\beta}^{\mathrm{vN}} G$.

For a C^*-algebra $B \subseteq \mathcal{B}(\mathcal{H})$ let $\mathcal{CB}_\sigma(B)$ be the space of completely bounded maps on B that extend to completely bounded, weak*-continuous maps on B''.

Definition 4.1. A C^*-dynamical system (A, G, α) will be called *weakly amenable* if there exists a net (F_i) of finitely supported Herz–Schur (A, G, α)-multipliers such that $F_i(t)$ is a finite rank completely bounded map on A for all $t \in G$,

$$F_i(t)(a) \xrightarrow{\|\cdot\|} a \quad \text{for all } t \in G,\ a \in A,$$

and $\sup_i \|F_i\|_{\mathrm{HS}} = K < \infty$. The infimum of all such K will be denoted by $\Lambda_{\mathrm{cb}}(A, G, \alpha)$.

A W^*-dynamical system (M, G, β), with $M \subseteq \mathcal{B}(\mathcal{H}_M)$, will be called *weakly amenable* if there is a net $F_i : G \to \mathcal{CB}_\sigma(M_\beta)$ of finitely supported Herz–Schur id-multipliers of (M_β, G, β) such that $F_i(t)$ extends to a finite rank map on M for all $t \in G$,

$$F_i(t)(a) \xrightarrow{\mathrm{w}^*} a \quad \text{for all } t \in G,\ a \in M, \tag{3}$$

and $\sup_i \|F_i\|_{\mathrm{HS}} = K < \infty$.

Observe that if $A = \mathbb{C}$ then the finite rank condition is always satisfied, so Definition 4.1 reduces to weak amenability of G.

Remark 4.2. *If (A, G, α) is a weakly amenable C^*-dynamical system with A unital such that the maps F_i of Definition* 4.1 *satisfy*

$$F_i(t) \circ \alpha_r = \alpha_r \circ F_i(t), \quad r, t \in G, \tag{4}$$

then G is weakly amenable.

Proof. Suppose that A is faithfully represented on a separable Hilbert space $\mathcal{H}$ and that (A, G, α) is weakly amenable. Take a net (F_i) of Herz–Schur (A, G, α)-multipliers satisfying the definition. Let $\xi \in \mathcal{H}$ be a unit vector. Condition (4) ensures that the map

$$v_i : G \to \mathbb{C}, \quad v_i(ts^{-1}) := \langle \mathcal{N}(F_i)(s,t)(1_A)\xi, \xi \rangle, \quad s, t \in G,$$

is well-defined. Let V_i and W_i be the maps associated to $\mathcal{N}(F_i)$ in Theorem 2.3. Then

$$v_i(ts^{-1}) = \langle \mathcal{N}(F_i)(s,t)(1_A)\xi, \xi \rangle = \langle V_i(s)\xi, W_i(t)\xi \rangle, \quad s, t \in G.$$

Hence $v_i : G \to \mathbb{C}$ is a Herz–Schur multiplier (see Bożejko–Fendler [3]; these statements are part of the proof of [13, Proposition 4.1] for the particular case where (4) holds). Since F_i has finite support, so does v_i. We have

$$\|v_i\|_{\mathrm{M^{cb}}} \le \operatorname*{ess\,sup}_{s \in G} \|V_i(s)\| \operatorname*{ess\,sup}_{t \in G} \|W_i(t)\| = \|\mathcal{N}(F_i)\|_{\mathfrak{S}} = \|F_i\|_{\mathrm{HS}}.$$

Since

$$v_i(ts^{-1}) = \langle \mathcal{N}(F_i)(s,t)(1_A)\xi, \xi \rangle = \langle F_i(ts^{-1})(1_A)\xi, \xi \rangle \xrightarrow{i} \langle 1_A\xi, \xi \rangle = 1,$$

G is weakly amenable. ∎

We now prove our characterisation of weak amenability for C^*-dynamical systems. Since the reduced crossed product C^*-algebra and the collection of Herz–Schur (A, G, α)-multipliers do not depend on the representation θ of A, we will omit θ from our notation, working with a fixed representation of A on a separable Hilbert space $\mathcal{H}$.

Recall that a C^*-algebra B is said to have the CBAP if there exists a net (T_γ) of finite rank completely bounded maps on B such that $T_\gamma \xrightarrow{\gamma} \mathrm{id}_B$ in the point-norm topology and $\sup_\gamma \|T_\gamma\|_{\mathrm{cb}} = C < \infty$; the infimum of such constants C is denoted $\Lambda_{\mathrm{cb}}(B)$.

Theorem 4.3. *Let (A, G, α) be a C^*-dynamical system with G a discrete group and A a unital C^*-algebra. The following are equivalent:*

(i) *(A, G, α) is weakly amenable;*
(ii) *$A \rtimes_{\alpha,r} G$ has the completely bounded approximation property.*

Moreover, if the conditions hold then $\Lambda_{\mathrm{cb}}(A, G, \alpha) = \Lambda_{\mathrm{cb}}(A \rtimes_{\alpha,r} G)$.

Proof. (i)⇒(ii). Suppose that (F_i) is a net of Herz–Schur (A, G, α)-multipliers satisfying weak amenability of the system. It follows immediately that the net (S_{F_i}) of corresponding maps on $A \rtimes_{\alpha,r} G$ consists of completely bounded, finite rank maps satisfying $\sup \|S_{F_i}\|_{\mathrm{cb}} \le C < \infty$. It remains to show that $\|S_{F_i}(T)-T\| \xrightarrow{i} 0$ for all $T \in A \rtimes_{\alpha,r} G$. For this, it suffices to show that $\|S_{F_i}(\sum_t \pi(a_t)\lambda_t) - \sum_t \pi(a_t)\lambda_t\| \xrightarrow{i} 0$ when the sums are finite. Indeed, for any $T \in A \rtimes_{\alpha,r} G$ and $\epsilon > 0$, we can find $a_t \in A$ with $\|T - \sum_t \pi(a_t)\lambda_t\| < \epsilon$, where only a finite number of a_t are non-zero, so

$$\begin{aligned}\|S_{F_i}(T) - T\| &\leq \Big\|S_{F_i}(T) - S_{F_i}\Big(\sum_t \pi(a_t)\lambda_t\Big)\Big\| \\ &\quad + \Big\|S_{F_i}\Big(\sum_t \pi(a_t)\lambda_t\Big) - \sum_t \pi(a_t)\lambda_t\Big\| + \Big\|\sum_t \pi(a_t)\lambda_t - T\Big\| \\ &< C\epsilon + \Big\|S_{F_i}\Big(\sum_t \pi(a_t)\lambda_t\Big) - \sum_t \pi(a_t)\lambda_t\Big\| + \epsilon.\end{aligned}$$

Now

$$\begin{aligned}\Big\|S_{F_i}\Big(\sum_t \pi(a_t)\lambda_t\Big) - \sum_t \pi(a_t)\lambda_t\Big\| &= \Big\|\sum_t \pi(F_i(t)(a_t))\lambda_t - \sum_t \pi(a_t)\lambda_t\Big\| \\ &\leq \sum_t \|\pi(F_i(t)(a_t) - a_t)\lambda_t\| \xrightarrow{i} 0\end{aligned}$$

as $F_i(t)(a) \xrightarrow{i} a$ for all $a \in A$, $t \in G$. It follows that $\Lambda_{\mathrm{cb}}(A \rtimes_{\alpha,r} G) \leq \Lambda_{\mathrm{cb}}(A, G, \alpha)$.

(ii)⇒(i). We will use a similar idea to Haagerup [9, Lemma 2.5]. First consider a finite rank, completely bounded map $\rho : A \rtimes_{\alpha,r} G \to A \rtimes_{\alpha,r} G$. Take $T_1, \dots, T_k \in A \rtimes_{\alpha,r} G$ which span $\operatorname{ran}\rho$, so there are $\phi_1, \dots, \phi_k \in (A \rtimes_{\alpha,r} G)^*$ such that

$$\rho = \sum_{j=1}^k \phi_j \otimes T_j,$$

where $(\phi_j \otimes T_j)(T) = \phi_j(T)T_j$ $(T \in A \rtimes_{\alpha,r} G)$. We note that, for a matrix $(x_{p,q}) \in M_n(A \rtimes_{\alpha,r} G)$,

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^k \phi_j \otimes T_j\Big)^{(n)}(x_{p,q})\Big\| &\leq \sum_{j=1}^k \|(\phi_j \otimes T_j)^{(n)}(x_{p,q})\| \\ &= \sum_{j=1}^k \|\phi_j^{(n)}(x_{p,q}) \operatorname{diag}_n(T_j)\| \\ &\leq \sum_{j=1}^k \|\phi_j\| \, \|(x_{p,q})\| \, \|T_j\|,\end{aligned}$$

where $\operatorname{diag}_n(T)$ denotes the diagonal $n \times n$ matrix with each diagonal entry equal to T. Thus

$$\Big\|\sum_{j=1}^k \phi_j \otimes T_j\Big\|_{\mathrm{cb}} \leq \sum_{j=1}^k \|\phi_j\| \, \|T_j\|. \tag{5}$$

For each j and each $n \in \mathbb{N}$ find $a^i_{j,n} \in A$ and $s^i_{j,n} \in G$ such that $T_{j,n} := \sum_{i=1}^{k_{j,n}} \pi(a^i_{j,n})\lambda_{s^i_{j,n}}$ satisfies $\|T_j - T_{j,n}\| < 1/(nk \max_j \|\phi_j\|)$. Define $\rho_n := \sum_{j=1}^k \phi_j \otimes T_{j,n}$. Then

$$\begin{aligned}(6)\qquad \|\rho - \rho_n\|_{\mathrm{cb}} &= \Big\|\Big(\sum_{j=1}^k \phi_j \otimes T_j\Big) - \Big(\sum_{j=1}^k \phi_j \otimes T_{j,n}\Big)\Big\|_{\mathrm{cb}} \\ &\le \sum_{j=1}^k \|\phi_j \otimes (T_j - T_{j,n})\|_{\mathrm{cb}} \le \sum_{j=1}^k \|\phi_j\|\,\|T_j - T_{j,n}\| < \frac{1}{n}.\end{aligned}$$

Now let (ρ_γ) be a net of maps on $A \rtimes_{\alpha,r} G$ satisfying the conditions of the CBAP. By the above procedure we obtain a net of maps $(\rho'_{\gamma,n})$ on $A \rtimes_{\alpha,r} G$ which are finite rank, with range in $\operatorname{span}\{\pi(a)\lambda_t : a \in A,\ t \in G\}$. It is easily checked that $\rho'_{\gamma,n} \xrightarrow{\gamma,n} \mathrm{id}$ in point-norm, using the product directed set. As in (5) we observe that each $\rho'_{\gamma,n}$ is completely bounded; by (6) we have $\|\rho_\gamma - \rho'_{\gamma,n}\|_{\mathrm{cb}} < 1/n$ for all γ and all $n \in \mathbb{N}$, so $\|\rho'_{\gamma,n}\|_{\mathrm{cb}} < \|\rho_\gamma\|_{\mathrm{cb}} + 1/n$. Let $C = \sup_\gamma \|\rho_\gamma\|_{\mathrm{cb}}$ and define

$$\rho_{\gamma,n} := \frac{C}{C + 1/n}\rho'_{\gamma,n},$$

so that $(\rho_{\gamma,n})$ is a net satisfying the CBAP for $A \rtimes_{\alpha,r} G$, uniformly bounded by C, and with range in $\operatorname{span}\{\pi(a)\lambda_t : a \in A,\ t \in G\}$. Define $F_{\gamma,n} : G \to \mathcal{CB}(A)$ by

$$(7)\qquad F_{\gamma,n}(t)(a) := \mathcal{E}\big(\rho_{\gamma,n}(\pi(a)\lambda_t)\lambda_t^*\big), \qquad a \in A,\ t \in G.$$

It is easy to see that $\operatorname{supp} F_{\gamma,n} \subseteq \{s^i_{j,n} : 1 \le i \le k_{j,n},\ 1 \le j \le k\}$. As $\rho_{\gamma,n}$ is finite rank, with range spanned by finite sums of elements of the form $\pi(a)\lambda_r$ ($a \in A$, $r \in G$), it follows that each $F_{\gamma,n}(t)$ is a finite rank map on A, with $\operatorname{ran} F_{\gamma,n}(t) \subseteq \operatorname{span}\{a \in A : \pi(a)\lambda_t \in \operatorname{ran}\rho_{\gamma,n}\}$. Since $\rho_{\gamma,n} \xrightarrow{\gamma,n} \mathrm{id}$ in point-norm, for all $t \in G$, $a \in A$ we have

$$F_{\gamma,n}(t)(a) = \mathcal{E}\big(\rho_{\gamma,n}(\pi(a)\lambda_t)\lambda_t^*\big) \xrightarrow{\gamma,n} \mathcal{E}(\pi(a)\lambda_{tt^{-1}}) = a.$$

It remains to show that each $F_{\gamma,n}$ is a Herz–Schur (A, G, α)-multiplier and $\|S_{F_{\gamma,n}}\|_{\mathrm{cb}} = \|\rho_{\gamma,n}\|_{\mathrm{cb}}$. Write the completely bounded maps $\rho_{\gamma,n}$ as $\rho_{\gamma,n}(\cdot) = W^*_{\gamma,n}\Psi_{\gamma,n}(\cdot)V_{\gamma,n}$, where $V_{\gamma,n}, W_{\gamma,n}$ are bounded operators and $\Psi_{\gamma,n}$ is a representation. To see that $F_{\gamma,n}$ is a Herz–Schur (A, G, α)-multiplier calculate

$$\begin{aligned}\mathcal{N}(F_{\gamma,n})(s,t)(a) &= \alpha_{t^{-1}}\big(\mathcal{E}(\rho_{\gamma,n}(\pi(\alpha_t(a))\lambda_{ts^{-1}})\lambda_{st^{-1}})\big) \\ &= \alpha_{t^{-1}}\big(\mathcal{E}(\rho_{\gamma,n}(\lambda_t\pi(a)\lambda_{s^{-1}})\lambda_{st^{-1}})\big) \\ &= \mathcal{E}\big(\lambda_{t^{-1}}\rho_{\gamma,n}(\lambda_t\pi(a)\lambda_{s^{-1}})\lambda_s\big)\end{aligned}$$

$$= \mathcal{E}\big(\lambda_{t^{-1}} W^*_{\gamma,n} \Psi_{\gamma,n}(\lambda_t)\Psi_{\gamma,n}(\pi(a))\Psi_{\gamma,n}(\lambda_{s^{-1}})V_{\gamma,n}\lambda_s\big)$$
$$= U^*\lambda_{t^{-1}} W^*_{\gamma,n} \Psi_{\gamma,n}(\lambda_t)\Psi_{\gamma,n}(\pi(a))\Psi_{\gamma,n}(\lambda_{s^{-1}})V_{\gamma,n}\lambda_s U$$
$$= \mathcal{W}_{\gamma,n}(t)^*\Psi_{\gamma,n}(\pi(a))\mathcal{V}_{\gamma,n}(s),$$

where $U : \mathcal{H} \to \ell^2(G) \otimes \mathcal{H}$, $\xi \mapsto \delta_e \otimes \xi$, and

$$\mathcal{V}_{\gamma,n}(s) := \Psi_{\gamma,n}(\lambda_{s^{-1}})V_{\gamma,n}\lambda_s U, \quad \mathcal{W}_{\gamma,n}(t) := \Psi_{\gamma,n}(\lambda_{t^{-1}})W_{\gamma,n}\lambda_t U,$$

so $F_{\gamma,n}$ is a Herz–Schur (A, G, α)-multiplier by Theorem 2.3.

For the norm equality let $(e_l)_\Lambda$ be an orthonormal basis for $\mathcal{H}$, let

$$V : \ell^2(G) \otimes \mathcal{H} \to \ell^2(G) \otimes \ell^2(G) \otimes \mathcal{H}, \quad \delta_g \otimes e_l \mapsto \delta_g \otimes \delta_g \otimes e_l,$$

where $\{\delta_g : g \in G\}$ denotes the canonical orthonormal basis for $\ell^2(G)$, and let τ denote the coaction

$$\tau : A \rtimes_{\alpha,r} G \to C^*_r(G) \otimes_{\min} A \rtimes_{\alpha,r} G, \quad \pi(a)\lambda_t \mapsto \lambda^G_t \otimes \pi(a)\lambda_t,$$

for all $a \in A$, $t \in G$. We claim

$$S_{F_{\gamma,n}}(x) = V^*(\mathrm{id} \otimes \rho_{\gamma,n})\tau(x)V, \quad x \in A \rtimes_{\alpha,r} G, \tag{8}$$

which implies $S_{F_{\gamma,n}}$ is completely bounded, with $\|S_{F_{\gamma,n}}\|_{\mathrm{cb}} = \|\rho_{\gamma,n}\|_{\mathrm{cb}}$. To prove the claim we first assume $\rho_{\gamma,n}$ has one-dimensional range generated by $\pi(b)\lambda_r$ for some $b \in A$, $r \in G$. Then, for $x, y \in G$, $l, m \in \Lambda$,

$$\begin{aligned}
&\langle V^*(\mathrm{id} \otimes \rho_{\gamma,n})\tau(\pi(a)\lambda_t)V(\delta_x \otimes e_m), \delta_y \otimes e_l\rangle \\
&\qquad = \langle \lambda_t \otimes \rho_{\gamma,n}(\pi(a)\lambda_t)(\delta_x \otimes \delta_x \otimes e_m), \delta_y \otimes \delta_y \otimes e_l\rangle \\
&\qquad = \langle \delta_{tx}, \delta_y\rangle\langle \rho_{\gamma,n}(\pi(a)\lambda_t)(\delta_x \otimes e_m), \delta_y \otimes e_l\rangle \\
&\qquad = \langle \delta_{tx}, \delta_y\rangle\langle \pi(b)\lambda_r(\delta_x \otimes e_m), \delta_y \otimes e_l\rangle \\
&\qquad = \langle \delta_{tx}, \delta_y\rangle\langle \pi(b)\lambda_r(\delta_x \otimes e_m)(y), e_l\rangle \\
&\qquad = \langle \delta_{tx}, \delta_y\rangle\langle \alpha_{y^{-1}}(b)e_m, e_l\rangle\langle \delta_{rx}, \delta_y\rangle.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\langle S_{F_{\gamma,n}}(\pi(a)\lambda_t)(\delta_x \otimes e_m), \delta_y \otimes e_l\rangle \\
&\qquad = \langle \pi(F_{\gamma,n}(t)(a))\lambda_t(\delta_x \otimes e_m), \delta_y \otimes e_l\rangle \\
&\qquad = \big\langle \pi(\mathcal{E}(\rho_{\gamma,n}(\pi(a)\lambda_t)\lambda_{t^{-1}}))\lambda_t(\delta_x \otimes e_m), \delta_y \otimes e_l\big\rangle \\
&\qquad = \big\langle \pi(\mathcal{E}(\pi(b)\lambda_{rt^{-1}}))\lambda_t(\delta_x \otimes e_m), \delta_y \otimes e_l\big\rangle \\
&\qquad = \langle \delta_r, \delta_t\rangle\langle \pi(b)\lambda_t(\delta_x \otimes e_m), \delta_y \otimes e_l\rangle \\
&\qquad = \langle \delta_r, \delta_t\rangle\langle \alpha_{y^{-1}}(b)e_m, e_l\rangle\langle \delta_{tx}, \delta_y\rangle.
\end{aligned}$$

It follows that $V^*(\mathrm{id} \otimes \rho_{\gamma,n})\tau(\pi(a)\lambda_t)V = S_{F_{\gamma,n}}(\pi(a)\lambda_t)$. By linearity and continuity we obtain (8) when $\rho_{\gamma,n}$ has one-dimensional range. The linearity of the inner product then implies that (8) holds in the general case that $\rho_{\gamma,n}$ takes values in $\mathrm{span}\{\pi(b_i)\lambda_{r_i} : i = 1, \dots, k\}$. The equality $\|S_{F_{\gamma,n}}\|_{\mathrm{cb}} =$

$\|\rho_{\gamma,n}\|_{\mathrm{cb}}$ follows, so $(F_{\gamma,n})$ is a net satisfying weak amenability of (A, G, α). It also follows that $\Lambda_{\mathrm{cb}}(A, G, \alpha) \le \Lambda_{\mathrm{cb}}(A \rtimes_{\alpha,r} G)$. ∎

REMARK 4.4. The constant Λ_{cb} introduced in Definition 4.1 reduces to the familiar constants defined in Section 1 in degenerate cases. Indeed, if G is a discrete group such that the system $(\mathbb{C}, G, 1)$ is weakly amenable then G is weakly amenable by Remark 4.2 or Theorem 4.3; moreover, by Theorem 4.3,

$$\Lambda_{\mathrm{cb}}(\mathbb{C}, G, 1) = \Lambda_{\mathrm{cb}}(\mathbb{C} \rtimes_{1,r} G) = \Lambda_{\mathrm{cb}}(C^*_r(G)) = \Lambda_{\mathrm{cb}}(G).$$

Similarly, if the C^*-dynamical system $(A, \{e\}, 1)$ is weakly amenable then

$$\Lambda_{\mathrm{cb}}(A, \{e\}, 1) = \Lambda_{\mathrm{cb}}(A \rtimes_{1,r} \{e\}) = \Lambda_{\mathrm{cb}}(A).$$

In fact, Sinclair–Smith [15, Theorem 3.4] have shown that for an amenable discrete group G, $\Lambda_{\mathrm{cb}}(A \rtimes_{\alpha,r} G) = \Lambda_{\mathrm{cb}}(A)$, so when (A, G, α) is a discrete C^*-dynamical system with G amenable, we have

$$\Lambda_{\mathrm{cb}}(A, G, \alpha) = \Lambda_{\mathrm{cb}}(A \rtimes_{\alpha,r} G) = \Lambda_{\mathrm{cb}}(A).$$

We now turn to characterising weak amenability of W^*-dynamical systems.

LEMMA 4.5. *Let (M, G, β) be a W^*-dynamical system, with G a discrete group, and (F_i) a net of Herz–Schur* id*-multipliers of the underlying C^*-dynamical system (M_β, G, β). The following are equivalent:*

(i) *$F_i(t)(a) \xrightarrow{i} a$ in the weak* topology for all $t \in G$, $a \in M$ (equation* (3) *above);*
(ii) *$s_{F_i} u \xrightarrow{i} u$ in $\mathcal{A}^{\mathrm{id}}(M, G, \beta)$ for all $u \in \mathcal{A}^{\mathrm{id}}(M, G, \beta)$.*

Proof. (i)⇒(ii). By Remark 3.2 finitely supported functions are dense in $\mathcal{A}^{\mathrm{id}}(M, G, \beta)$, so it suffices to prove the claim for singly supported $u \in \mathcal{A}^{\mathrm{id}}(M, G, \beta)$. Suppose $u \in \mathcal{A}^{\mathrm{id}}(M, G, \beta)$ is supported on $\{s\}$ and $u(t)(a) = \sum_{n=1}^\infty \langle \pi(a)\lambda_t \xi_n, \eta_n\rangle$ $(t \in G,\, a \in M)$ for some families satisfying $\sum_{n=1}^\infty \|\xi_n\|^2 < \infty$ and $\sum_{n=1}^\infty \|\eta_n\|^2 < \infty$. Since λ_s is an isometry, it follows that the functional in $\pi(M)_*$ given by $\pi(a) \mapsto \sum_{n=1}^\infty \langle \pi(a)\lambda_s \xi_n, \eta_n\rangle$ has the same norm as u; thus $\|u(s)\| = \|u\|_{\mathcal{A}}$. Since $s_{F_i} u$ is also supported on $\{s\}$, we have

$$\|s_{F_i} u - u\|_{\mathcal{A}} = \|u(s) \circ F_i(s) - u(s)\| = \sup_{\|a\| \le 1} |u(s)(F_i(s)(a) - a)| \xrightarrow{i} 0.$$

Condition (ii) follows.

(ii)⇒(i). For any $a \in A$, $t \in G$ and $u \in \mathcal{A}^{\mathrm{id}}(M, G, \beta)$,

$$|\langle \pi(F_i(t)(a))\lambda_t - \pi(a)\lambda_t, u\rangle| = |\langle \pi(a)\lambda_t, s_{F_i} u\rangle - \langle \pi(a)\lambda_t, u\rangle| \xrightarrow{i} 0,$$

so $u(t)(F_i(t)(a)) \xrightarrow{i} u(t)(a)$. As u varies, $u(t)$ can take any value in M_*; thus $F_i(t)(a)$ converges to a in the weak* topology. ∎

THEOREM 4.6. *Let G be a discrete group, $M \subseteq \mathcal{B}(\mathcal{H}_M)$ a von Neumann algebra acting on a separable Hilbert space, and (M, G, β) a W^*-dynamical system. Consider the conditions:*

(i) *(M, G, β) is weakly amenable;*
(ii) *$M \rtimes_\beta^{\mathrm{vN}} G$ has the weak* completely bounded approximation property.*

Then (i)⇒(ii). *If G is weakly amenable then* (i)⇒(ii).

Proof. (i)⇒(ii). Suppose that (F_i) is a net of Herz–Schur id-multipliers of the underlying C^*-dynamical system (M_β, G, β) satisfying Definition 4.1. Then the associated net (S_{F_i}) of maps on $M \rtimes_\beta^{\mathrm{vN}} G$ is completely bounded, weak*-continuous, and finite rank. Finally, using the identification of $(M \rtimes_\beta^{\mathrm{vN}} G)_*$ with $\mathcal{A}^{\mathrm{id}}(M, G, \beta)$, we find that for any $u \in \mathcal{A}^{\mathrm{id}}(M, G, \beta)$ and any $T \in M \rtimes_\beta^{\mathrm{vN}} G$,

$$\langle S_{F_i} T, u\rangle = \langle T, s_{F_i} u\rangle \xrightarrow{i} \langle T, u\rangle$$

by Lemma 4.5, so $S_{F_i} T$ converges to T in the weak* topology.

(ii)⇒(i). Suppose $M \rtimes_\beta^{\mathrm{vN}} G$ has the weak* CBAP. Given a finite set $E \subseteq G$, $\epsilon > 0$, and a collection $\Omega \subseteq M_*$, choose $\rho : M \rtimes_\beta^{\mathrm{vN}} G \to M \rtimes_\beta^{\mathrm{vN}} G$ such that

$$F : G \to \mathcal{CB}_\sigma(M_\beta), \quad F(t)(a) := \mathcal{E}(\rho(\pi(a)\lambda_t)\lambda_{t^{-1}}), \quad a \in M,\ t \in G, \tag{9}$$

satisfies $|\omega(a - F(t)(a))| < \epsilon$ for all $a \in M$, $t \in E$, $\omega \in \Omega$. In this way we produce a net (F_i), indexed by triples of the form (E, ϵ, Ω), such that $F_i(t)(a) \to a$ in the weak* topology. For each $t \in G$, $F(t)$ defined above is a finite rank map on M as in the proof of Theorem 4.3; indeed, suppose $\rho = \sum_{j=1}^k \phi_j \otimes T_j$, where ϕ_j is a functional and $T_j \in M \rtimes_\beta^{\mathrm{vN}} G$. Then

$$F(t)(a) = \mathcal{E}\big(\rho(\pi(a)\lambda_t)\lambda_{t^{-1}}\big) = \sum_{j=1}^k \phi_j(\pi(a)\lambda_t)\mathcal{E}(T_j\lambda_{t^{-1}}),$$

so that $\{\mathcal{E}(T_j\lambda_{t^{-1}}) : j = 1, \ldots, k\}$ spans $\operatorname{ran} F(t)$. Similar calculations to those in the proof of Theorem 4.3 show that $\|S_F\|_{\mathrm{cb}} = \|\rho\|_{\mathrm{cb}}$ and F is a Herz–Schur (M_β, G, β)-multiplier. Each S_F is a composition of weak*-continuous maps, so is weak*-extendable. Thus the net (F_i) satisfies all the conditions of weak amenability of (M, G, β) except that it may not be finitely supported. To correct this we use the assumption that G is weakly amenable. Let (φ_j) be a net of functions on G implementing weak amenability. Define another net, indexed by the product directed set,

$$F_{i,j} : G \to \mathcal{CB}_\sigma(M), \quad F_{i,j}(t)(a) := \varphi_j(t)F_i(t)(a), \quad t \in G,\ a \in M,$$

which is a net of Herz–Schur id-multipliers of (M_β, G, β) such that $S_{F_{i,j}} = S_{\varphi_j} \circ S_{F_i}$. From the properties of φ_j and F_i we deduce that each $F_{i,j}$ is finitely supported, $F_{i,j}(t)$ is finite rank for all $t \in G$, and $F_{i,j}(t)(a)$ converges to a in the weak* topology. Finally, $\|F_{i,j}\|_{\mathrm{HS}} = \|S_{F_{i,j}}\|_{\mathrm{cb}} \le \|S_{\varphi_j}\|_{\mathrm{cb}}\|S_{F_i}\|_{\mathrm{cb}}$, so the net is uniformly bounded. ∎

REMARKS 4.7. (I) In the proof of (ii)⇒(i) above we required weak amenability of G; to see why this requirement arose let us return to the proof of Theorem 4.3. There we are able to approximate in norm the operators ρ_γ, which implement the CBAP of $A \rtimes_{\alpha,r} G$, by operators $\rho_{\gamma,n}$ with finite-dimensional range spanned by elements of the form $\pi(a)\lambda_t$, such that $\|\rho_{\gamma,n}\|_{\mathrm{cb}}$ is closely related to $\|\rho_\gamma\|_{\mathrm{cb}}$; these estimates allowed us to identify the support and Herz–Schur norm of $F_{\gamma,n}$. Such norm estimates are not available in the setting of Theorem 4.6, so the extra hypothesis seems to be required to use the techniques in this paper.

(II) If in the above proof we make the stronger assumption $\Lambda_{\mathrm{cb}}(G) = 1$ then the net $(\varphi_{i,n})$ may be chosen such that $\|S_{\varphi_{i,n}}\|_{\mathrm{cb}}$ is uniformly bounded by 1. With this assumption on G we obtain $\Lambda^{\mathrm{vN}}_{\mathrm{cb}}(M,G,\beta) \le \Lambda_{\mathrm{cb}}(M \rtimes^{\mathrm{vN}}_\beta G)$, where $\Lambda^{\mathrm{vN}}_{\mathrm{cb}}$ is the natural weak amenability constant of a W^*-dynamical system. It follows that if $\Lambda_{\mathrm{cb}}(G)=1$ then we have $\Lambda^{\mathrm{vN}}_{\mathrm{cb}}(M,G,\beta)=\Lambda_{\mathrm{cb}}(M \rtimes^{\mathrm{vN}}_\beta G)$. It would be interesting to have a characterisation of when these two weak amenability constants coincide.

Suppose that (A, G, α) is a C^*-dynamical system with G an amenable discrete group and A a nuclear C^*-algebra. It is well-known (e.g. Brown–Ozawa [4, Theorem 4.2.6]) that this implies $A \rtimes_{\alpha,r} G$ is nuclear. It is natural to ask if this fact persists for weak amenability and the CBAP: do the CBAP for A and weak amenability of G imply that $A \rtimes_{\alpha,r} G$ has the CBAP? Haagerup–Kraus [10, Remark 3.10] give an example of a W^*-dynamical system showing that in general this is not true, which we reproduce here as a C^*-dynamical system. Both $\mathrm{SL}(2,\mathbb{Z})$ and $\mathbb{Z}^2$ are weakly amenable, but their semidirect product $\mathbb{Z}^2 \rtimes_\mu \mathrm{SL}(2,\mathbb{Z})$ is not [10, p. 670] (μ denotes the usual action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{Z}^2$). Since the C^*-algebras $C^*_r(\mathbb{Z}^2) \rtimes_{\mu,r} \mathrm{SL}(2,\mathbb{Z})$ and $C^*_r(\mathbb{Z}^2 \rtimes_\mu \mathrm{SL}(2,\mathbb{Z}))$ are isomorphic, it follows that the crossed product of a C^*-algebra with the CBAP by a weakly amenable group need not have the CBAP.

Sinclair–Smith [15] have shown that if G is amenable and A has the CBAP then $A \rtimes_{\alpha,r} G$ has the CBAP. To finish this paper we give an example of an additional assumption under which this implication can be recovered for weakly amenable groups.

PROPOSITION 4.8. *Let (A, G, α) be a C^*-dynamical system with G a discrete group. The following are equivalent:*

(i) *G is weakly amenable, A has the CBAP and the approximating maps $\phi_i : A \to A$ satisfy $\phi_i \circ \alpha_t = \alpha_t \circ \phi_i$ for all $t \in G$ and all i;*

(ii) *the system (A,G,α) is weakly amenable and for all $r,t\in G$ and all i the approximating Herz–Schur (A,G,α)-multipliers $F_i : G\to \mathcal{CB}(A)$ satisfy $F_i(t)(\alpha_r(a)) = \alpha_r(F_i(t)(a))$.*

Proof. (i)⇒(ii). The condition on the maps (ϕ_i) implies that the map

$$\tilde{\phi}_i : A\rtimes_{\alpha,r} G\to A\rtimes_{\alpha,r} G,\qquad \sum_t \pi(a_t)\lambda_t\mapsto \sum_t \pi(\phi_i(a_t))\lambda_t,\quad a_t\in A,\, t\in G,$$

can be identified with the restriction of $I_{\ell^2(G)}\otimes\phi_i$ on $\mathcal{B}(\ell^2(G))\otimes_{\min} A$ to $A\rtimes_{\alpha,r} G$. It follows from [6, Lemma 1.5] that $\tilde{\phi}_i$ is completely bounded and $\|\tilde{\phi}_i\|_{\mathrm{cb}}\le\|\phi_i\|_{\mathrm{cb}}$. Let (v_γ) be a net of scalar-valued functions on G satisfying weak amenability of G and let S_{v_γ} be the completely bounded map on $A\rtimes_{\alpha,r} G$ associated to the (classical) Herz–Schur multiplier v_γ as in [13, Proposition 4.1]. Denote by $S_{\gamma,i}$ the composition $S_{v_\gamma}\circ\tilde{\phi}_i$. These implement the CBAP for $A\rtimes_{\alpha,r} G$; indeed, if $\sup_i\|\phi_i\|_{\mathrm{cb}}\le C_1$ and $\sup_\gamma\|v_\gamma\|_{\mathrm{M^{cb}}}\le C_2$ then $\sup\|S_{\gamma,i}\|_{\mathrm{cb}}\le C_1C_2$, each $S_{\gamma,i}$ is finite rank, and for any $T\in A\rtimes_{\alpha,r} G$,

$$\begin{aligned}\|S_{\gamma,i}(T)-T\| &\le \|S_{v_\gamma}(\tilde{\phi}_i(T))-S_{v_\gamma}(T)\|+\|S_{v_\gamma}(T)-T\|\\ &\le C_2\|\tilde{\phi}_i(T)-T\|+\|S_{v_\gamma}(T)-T\|\xrightarrow{\gamma,i}0.\end{aligned}$$

It follows from Theorem 4.3 that the system (A,G,α) is weakly amenable. To prove the covariance condition we first calculate the form of the Herz–Schur (A,G,α)-multipliers defined in the proof of Theorem 4.3:

$$F_{\gamma,i}(t)(a) := \mathcal{E}(S_{\gamma,i}(\pi(a)\lambda_t)\lambda_t^*) = \mathcal{E}\big(\pi(v_\gamma(t)\phi_i(a))\big) = v_\gamma(t)\phi_i(a).$$

Thus, for any $r\in G$,

$$\alpha_r(F_{\gamma,i}(t)(a)) = v_\gamma(t)\alpha_r(\phi_i(a)) = v_\gamma(t)\phi_i(\alpha_r(a)) = F_{\gamma,i}(t)(\alpha_r(a)).$$

(ii)⇒(i). Let (F_i) be a net of Herz–Schur (A,G,α)-multipliers satisfying weak amenability of the system and the covariance condition. Weak amenability of G follows as in Remark 4.2. Define

$$\phi_i : A\to A,\qquad a\mapsto \mathcal{E}(S_{F_i}(\pi(a))),\quad a\in A,$$

to obtain a net of maps easily seen to satisfy the CBAP for A. Now calculate

$$\begin{aligned}\phi_i(\alpha_t(a)) &= \mathcal{E}\big(S_{F_i}(\pi(\alpha_t(a)))\big) = \mathcal{E}\big(\pi(F_i(e)(\alpha_t(a)))\big) = \mathcal{E}\big(\pi(\alpha_t(F_i(e)(a)))\big)\\ &= \alpha_t(F_i(e)(a)) = \alpha_t\big(\mathcal{E}(S_{F_i}(\pi(a)))\big) = \alpha_t(\phi_i(a)),\end{aligned}$$

as required. ■

Acknowledgements. My sincere thanks to my advisor Ivan Todorov for his guidance during this work. I would also like to thank the EPSRC for funding my PhD position. Finally, I thank the anonymous referee for a number of helpful comments and corrections which improved the paper.

References

[1] E. Bédos and R. Conti, *Fourier series and twisted* C*-*crossed products*, J. Fourier Anal. Appl. 21 (2015), 32–75.

[2] E. Bédos and R. Conti, *The Fourier–Stieltjes algebra of a C^*-dynamical system*, Int. J. Math. 27 (2016), no. 6, art. 1650050, 50 pp.

[3] M. Bożejko and G. Fendler, *Herz–Schur multipliers and completely bounded multipliers of the Fourier algebra of a locally compact group*, Boll. Un. Mat. Ital. A (6) 3 (1984), 297–302.

[4] N. P. Brown and N. Ozawa, *C^*-Algebras and Finite-Dimensional Approximations*, Grad. Stud. Math. 88, Amer. Math. Soc., Providence, RI, 2008.

[5] M. Cowling and U. Haagerup, *Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one*, Invent. Math. 96 (1989), 507–549.

[6] J. De Cannière and U. Haagerup, *Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups*, Amer. J. Math. 107 (1985), 455–500.

[7] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France 92 (1964), 181–236.

[8] M. Fujita, *Banach algebra structure in Fourier spaces and generalization of harmonic analysis on locally compact groups*, J. Math. Soc. Japan 31 (1979), 53–67.

[9] U. Haagerup, *Group C^*-algebras without the completely bounded approximation property*, J. Lie Theory 26 (2016), 861–887.

[10] U. Haagerup and J. Kraus, *Approximation properties for group C^*-algebras and group von Neumann algebras*, Trans. Amer. Math. Soc. 344 (1994), 667–699.

[11] S. Knudby, *Approximation properties for groups and von Neumann algebras*, Ph.D. thesis, Dept. of Math. Sci., Univ. of Copenhagen, 2014; http://www.math.ku.dk/noter/filer/phd14sk.pdf.

[12] H. Leptin, *Sur l'algèbre de Fourier d'un groupe localement compact*, C. R. Acad. Sci. Paris Sér. A-B 266 (1968), A1180–A1182.

[13] A. McKee, I. G. Todorov and L. Turowska, *Herz–Schur multipliers of dynamical systems*, Adv. Math. 331 (2018), 387–438.

[14] G. K. Pedersen, *C^*-Algebras and Their Automorphism Groups*, London Math. Soc. Monogr. 14, Academic Press, London, 1979.

[15] A. M. Sinclair and R. R. Smith, *The completely bounded approximation property for discrete crossed products*, Indiana Univ. Math. J. 46 (1997), 1311–1322.

[16] H. Takai, *On a Fourier expansion in continuous crossed products*, Publ. RIMS Kyoto Univ. 11 (1976), 849–880.

[17] M. Takesaki, *Theory of Operator Algebras II*, Encyclopaedia Math. Sci. 125, Springer, Berlin, 2003.

[18] D. P. Williams, *Crossed Products of C^*-Algebras*, Math. Surveys Monogr. 134, Amer. Math. Soc., Providence, RI, 2007.

Andrew McKee
Department of Mathematical Sciences
Chalmers University of Technology
and the University of Gothenburg
SE-412 96 Göteborg, Sweden
E-mail: amckee240@qub.ac.uk

Current address:
Faculty of Mathematics
University of Białystok
K. Ciołkowskiego 1M
15-245 Białystok, Poland